GROUP ALGEBRAS AND THE ARTIN-WEDDERBURN THEOREM

The first part of these notes is a short introduction to group algebras and their relationship to representation theory of finite groups. The latter part of these notes is dedicated to an outline of a proof of the Artin-Wedderburn Theorem, which was used heavily in the proof of the Double Centralizer Theorem on page 26. As some of us may not have familiarity with these concepts (or even need a review), the hope is that these notes will cover the basic ideas behind these concepts. By no means are these notes a substitution for the "real deal". Disclaimer: these notes are heavily inspired by Jack Jeffries MATH 901 notes of the Fall of 2021, which, among many other things, includes a nice and more thorough account of the topics discussed below.

1. Group Algebras and their Relationship to Representation Theory

Definition 1.1. Let R be a ring and G be any group. The group ring $R[G]$ is defined as

$$R[G] = \bigoplus_{g\in G} R\cdot g = \left\{ \sum_{g\in G} r_g \cdot g \mid r_g = 0 \text{for all but finitely many } g \in G \right\},$$

with addition and multiplication defined as follows: let $\sum_{g\in G} r_g \cdot g$ and $\sum_{g\in G} s_g \cdot g$ be elements in $R[G]$ and set

$$\sum_{g\in G} r_g \cdot g + \sum_{g\in G} s_g \cdot g := \sum_{g\in G} (r_g + s_g)\cdot g$$

$$\left(\sum_{g\in G} r_g \cdot g\right)\cdot\left(\sum_{g\in G} s_g \cdot g\right) := \sum_{g,h\in G} (r_g s_h)\cdot gh.$$

Exercise 1. *Show that $R[G]$ is commutative if and only if R is commutative and G is abelian.*

Exercise 2 (Universal Mapping Property of group rings)**.** *Let R, A be rings and G a group. Given a ring homomorphism $\psi : R \to A$ and a group homomorphism $\phi : G \to (A^\times, \cdot)$, such that for every $r \in R, g \in G$ we have that $\psi(r)$ and $\phi(g)$ commute in $(A, \cdot)$, show there is a unique ring homomorphism $\alpha : R[G] \to A$ such that $\alpha|_R = \psi$ and $\alpha|_G = \phi$.*

The following theorem illustrates the bridge between group algebras and representations.

Theorem 1.1. *Let K be a field, V a K-vector space, and G a group. There is a bijection*

$$\left\{ \begin{array}{c} K\text{-linear representations} \\ \text{of } G \text{ on } V \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} K[G]\text{-module structures on } V \\ \text{that extend the given action of } K \end{array} \right\}.$$

Moreover, if V and W are representations of G, then $\psi : V \to W$ is G-equivariant if and only if it is $K[G]$-linear.

Theorem 1.2 (Maschkey's Theorem)**.** *Let K be a field and G a finite group such that the characteristic of K does not divide $|G|$. Then $K[G]$ is left semi-simple.*

2. Artin-Wedderburn

We begin this section with three exercises, two of which will be used in the proof of the Artin-Wedderburn Theorem.

Exercise 3. *Let R be any ring and V and W left semi-simple R-modules. Show*

$$\mathrm{Hom}_R(V, W) = \begin{cases} D, V \cong W \\ 0, V \ncong W \end{cases},$$

where $D = \mathrm{End}_R(V)$ is a division ring.

Exercise 4. *Let R be a ring and M a left R-module. Show that the map*

$$\mathrm{End}_R(M^{\oplus n}) \xrightarrow{\Theta} \mathrm{Mat}_n(\mathrm{End}_R(M))$$
$$\phi \mapsto (\pi_i \circ \phi \circ \iota_j)_{i,j},$$

where π_i and ι_j denote the natural projection and inclusion maps, respectively.

Exercise 5. *Let R be any ring, M a left R-module, D a division ring, and $n \in \mathbb{N}$. Show:*

(1) $(R^{\mathrm{op}})^{\mathrm{op}} \cong R$, where R^{op} is the opposite ring of R.
(2) $(M^{\mathrm{op}})^{\mathrm{op}} \cong M$.
(3) $\mathrm{Mat}_n(D)^{\mathrm{op}} \cong \mathrm{Mat}_n(D^{\mathrm{op}})$.

Exercise 6. *Let D be a division ring. Show that $\mathrm{End}_{\mathrm{Mat}_n(D)}(D^n) \cong D^{\mathrm{op}}$, where D^n is the simple module of column vectors of size n.*

We first state the group ring version of Artin-Wedderburn and then state and prove a more general version of it. Disclosure: the statements below as well as the proof strategy are right out of Jack's Notes.

Theorem 2.1. *If G is a finite group and K is a field such that $\mathrm{char}(K) \nmid |G|$, then there is an isomorphism of rings*

$$K[G] \cong \mathrm{Mat}_{n_1}(D_1) \times \cdots \times \mathrm{Mat}_{n_m}(D_m),$$

where $D_1, \ldots, D_m$ are division rings. Furthermore, each D_i contains K (up to isomorphism) as a subring of its center and the above isomorphism is K-linear. In particular, $\dim_K(D_i) < \infty$.

Moroever, we have:

(1) m is the number of irreducible k-linear representation of G (up to isomorphism),
(2) the D_i's are the opposite rings of the endomorphism rings of these representations,
(3) the n_j's give the number of times each irreducible representation occurs in the decomposition of the regular representation of G,
(4) the numbers $n_1 \cdot \dim_k(D_1), \ldots, n_m \cdot \dim_k(D_m)$ give the dimensions of these representations, and

(5) $n_1^2 \cdot \dim_k(D_1) + \cdots + n_m^2 \cdot \dim_k(D_m) = |G|$.

Theorem 2.2 (Artin-Wedderburn Theorem (general version))**.** *Let R be a left semisimple ring. Then for some $m \geqslant 0$, positive integers $n_1, \ldots, n_m$, and division rings $D_1, \ldots, D_m$, there is a ring isomorphism*

$$R \cong \mathrm{Mat}_{n_1}(D_1) \times \cdots \times \mathrm{Mat}_{n_m}(D_m).$$

Moreover,

(1) m is the number of isomorphism classes of simple left R-modules.
(2) Say $M_1, \ldots, M_m$ are simple modules forming a complete set of representatives of these isomorphism classes. Then, after reordering, $D_i \cong \mathrm{End}_R(M_i)^{\mathrm{op}}$ and
(3) n_j is the number of times summands isomorphic to M_j occur in the decomposition of R into a direct sum of simple left modules.

Moreover, the data $(m; n_1, \ldots, n_m; D_1, \ldots, D_m)$ is unique up to a permutation of $\{1, \ldots, m\}$ and isomorphisms of division rings.

Proof. Write $R = M_1^{\oplus n_1} \oplus \cdots \oplus M_m^{\oplus n_m}$, where the M_i's are pairwise non-isomorphic left R-modules.

(1) Show that

$$\mathrm{End}_R(R) \cong \prod \mathrm{Mat}_{n_i}(\mathrm{End}_R(M_i)).$$

(2) Set $D_i' := \mathrm{End}_R(M_i)$. Show this is a division ring.
(3) With D_i' as above, we have

$$R^{\mathrm{op}} \cong \mathrm{Mat}_{n_1}(D_1') \times \cdots \times \mathrm{Mat}_{n_m}(D_m').$$

Show that

$$\left(\prod \mathrm{Mat}_{n_i}(D_i')\right)^{\mathrm{op}} \cong \prod \mathrm{Mat}_{n_i}(D_i).$$

(4) Conclude the existence of such a decomposition.
(5) For uniqueness see Jack's Notes

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