EXERCISES 10/29/2024

These exercises are of varying difficulty. If your group is stuck on a problem, I suggest trying the others first and then go back. I don't expect every group to finish this in our meeting, so if you like, you may work on these during your own time. **Exercises with a * are used later in the text.**

Please feel free to work on past problems, as well!

(1) (Exercise 3 pg. 25): Let $\rho : G \to \mathrm{GL}(V)$ be an irreducbile representation and assume that $\mathrm{End}_G(V) = K$. Denote by V^n the G-module $\underbrace{V \oplus \cdots \oplus V}_{n-\text{times}}$, which we identify with $V \otimes K^n$.
  - (a) Show that $\mathrm{End}_G(V^n) \cong M_n(K)$ in a "natural" way.
  - (b) Show that every G-submodule of V^n is of the form $V \otimes U$, where U is a subspace of K^n.
  - (c) Suppose $\mu : H \to \mathrm{GL}(W)$ is an irreducbile representation of a group H. Show that $V \otimes_K W$ is a simple $G \times H$-module.
  - (d) Show that $\langle G \rangle = \mathrm{End}(V)$. Here $\langle G \rangle$ is the image of G under $\rho : G \to \mathrm{GL}(V) \subset \mathrm{End}(V)$. Check the text for a hint.
  - (e) For every field extension l/K the representation of G on $V_L := V \otimes_K L$ is irreducible. Check the text for a hint.

(2) (Exercise 4 pg. 27): Let V be an irreducible finite dimensional representation of a group G, where $\mathrm{End}_G(V) = K$, and let W be an arbitrary finite dimensional representation of G.
  - (a) Consider linear map $\gamma : \mathrm{Hom}_G(V, W) \otimes V \to W$, for which $\alpha \otimes v \mapsto \alpha(v)$. Show that γ is injective, show that γ is G-invariant, and show that the image of γ is the *isotypic submodule* of W of type V (i.e, the sum of all the simple submodules of W isomorphic to V).
  - (b) If there is another group H acting on W, which commutes with the action of G on W, show that γ is also H-invariant.
  - (c) Assume that K is algebraically closed. Show that every simple $G \times H$-module is of the form $V \otimes U$ where V is a simple G-module and U is a simple H-module.
  - (d) (A question that Taylor has): Does the previous exercise hold if K is replaced by a field that is not algebraically closed?

(3) Show that the isotypic componnet in $V^{\otimes m}$ of the trivial representation of S_m is the symmetric power $S^m(V)$.

(4) If $\dim(V) \geqslant m$, show that every irreducible representation of S_m occurs in $V^{\otimes} m$.

(5) Let $\rho : G \to \mathrm{GL}(V)$ be a completely reducible representation. For any field extension K'/K, show that the representation of G on $V \otimes_K K'$ is completely reducible, as well. See hint in text.

(6) Let V be an irreducible finite dimensional K-representation of a group G. Show that $\mathrm{End}_G(V) = K$ if and only if $V \otimes_K K'$ is irreducible for every field extension K'/K. See hint in text.