EXERCISES 10/01/2024

These exercises are of varying difficulty. If your group is stuck on a problem, I suggest trying the others first and then go back. I don't expect every group to finish this in our meeting, so if you like, you may work on these during your own time. **Exercises with a * are used later in the text.**

Please feel free to work on past problems, as well!

(1) Exercises 4 pg. 4 and 17 pg. 8: Let V be a K-vector space of dimension n. Let $\mathrm{GL}_n(K)$ act on V in the usual way. Let $T_n \subset \mathrm{GL}_n$ be the subgroup of invertible diagonal matrices. If we choose the standard basis for V and the corresponding dual basis in V^*, we can identify the coordinate ring $K[V \oplus V^*]$ with $K[x_1, \ldots, x_n, z_1, \ldots, z_n]$.
   (a) Show that $K[V \oplus V^*]^{T_n} = K[x_1 z_1, \ldots, x_n z_n]$.
   (b) Let $T_n' \subset T_n$ be the subgroup of diagonal matrices with determinant one. What is $K[V \oplus V^*]^{T_n'}$?
   (c) Show that $K[V \oplus V^*]^{\mathrm{GL}_n(K)} = K[q]$, where q is the bilinear form defined by $q = x_1 z_1 + \cdots + x_n z_n$. *Hint*: The subset $Z := \{(v, \phi) \mid \phi(v) \neq 0\}$ of $V \oplus V^*$ is Zariski-Dense. Fix a pair(v_0, ϕ_0) such that $\phi_0(v_0) = 1$. Then for every $(v, \phi) \in Z$, there is a $g \in \mathrm{GL}(V)$ such that $g(v, \phi) = (v_0, \lambda\phi_0)$, where $\lambda = \phi(v)$.

(2) Exercise 1 pg.18: Show that every matrix $C \in M_{q\times p}$ of rank less than or equal to n can be written as a product $C = AB$ with a $(q \times n)$-matrix A and a $(n \times p)$-matrix B.

(3) Exercise 2 pg. 18: Show that for any n the set of $p \times q$ matrices of rank less than or equal to n forms a closed subvariety of $M_{p\times q}$. In other words it is the set of zeroes of some polynomials. *Hint*: Consider the $(n + 1 \times n + 1)$-minors.

(4) Exercise 3 pg. 18: Let $\rho : G \to \mathrm{GL}(W)$ be a finite dimensional representation of a group G. Then the ring of invariant $K[W]^G$ is normal. That is, $K[W]^G$ is integrally closed in its field of fractions.

(5) Exercise 4 pg 20: Show that the set of diagonalizable matrices is Zariski-dense in $M_n(K)$. *Hint*: For an algebraically closed field K this is a consequence of the Jordan Decomposition Theorem. For the general case, use 1.3 Exercise 13 and Remark 1.3.